Synthetic USD on Tezos

October 1, 2021

1. The Ecosystem from the Mathematical Finance Viewpoint

1.1. Variables.

(i) *time axis*

Variable	Description	Default Value(s)
t	discrete time index for the current time in seconds since inception	$\mathbb{N}_0$
T	typically some time instance in the future	$\mathbb{N}_0$

(ii) *minting*

Variable	Description	Default Value(s)
λ_1	target collateral level	300%
λ_0	emergency collateral level	200%
$\mathcal{J}_t$	set of all vaults existing in the ecosystem at time t	
$C_t^{(j)}$	posted collateral in XTZ at time t in the vault $j \in \mathcal{J}_t$	
$\mathbf{C}_t = \sum_{j \in \mathcal{J}_t} C_t^{(j)}$	total value of posted collateral in XTZ at time t	
$M_t^{(j)}$	number of minted uUSD coins at time t corresponding to the vault $j \in \mathcal{J}_t$	
$\mathbf{M}_t = \sum_{j \in \mathcal{J}_t} M_t^{(j)}$	total number of minted uUSD coins at time t	

(iii) *circulating uUSD coins*

Variable	Description	Default Value(s)
$\mathcal{L}_t$	set of all wallets for stable tokens in the ecosystem at time t	
$A_t^{(\ell)}$	number of uUSD coins at time t in the wallet $\ell \in \mathcal{L}_t$	

(iv) *conversion rates*

Variable	Description	Default Value(s)
S_t	value of 1 XTZ denominated in USD	
$f_t = \frac{S_t \mathbf{C}_t}{\mathbf{M}_t}$	coverage ratio of the collateral	
R_t	value of 1 uUSD denominated in XTZ	$\approx \frac{\min\{f_t, 1\}}{S_t}$
$\widetilde{R}_t$	value of 1 uUSD denominated in USD	$\approx R_t S_t$
$\overline{\Delta}$	cap for maximal foreign exchange (FX) deviance	0.25

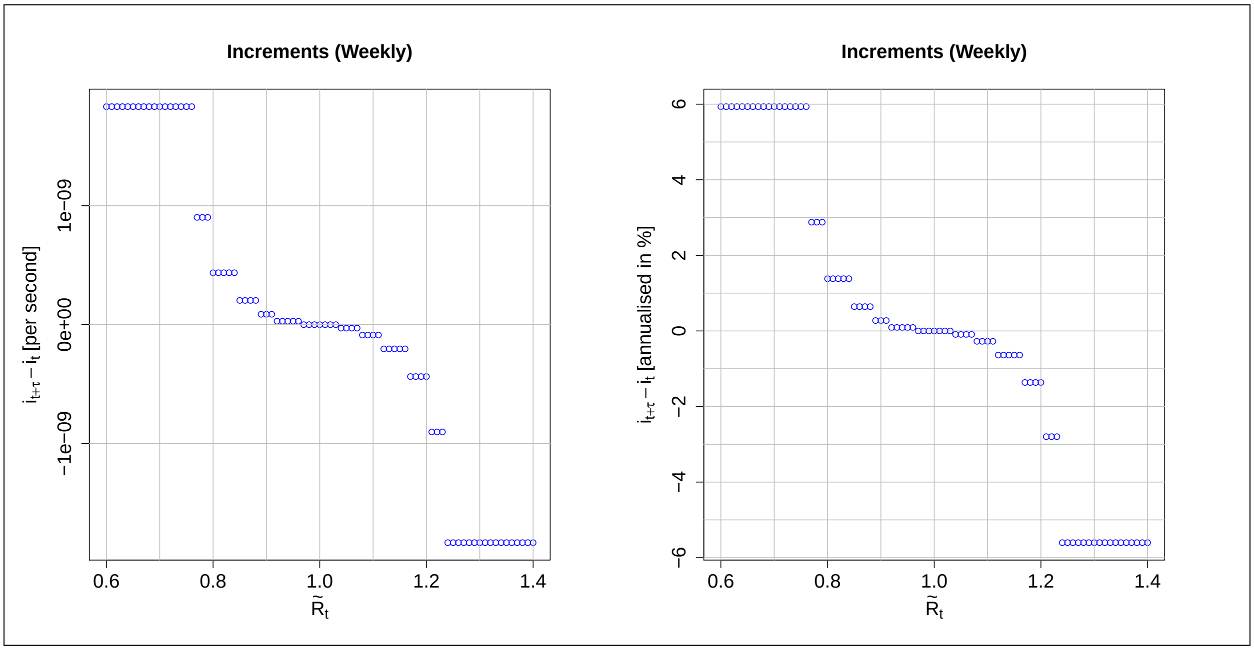
(v) *compounding factors*

Variable	Description	Default Value(s)
u_t	prevailing interest rate in the USD market per second	
$U(t,T) = \prod_{s=t}^{T-1}(1+u_s)$	compounding factor in USD between t and T	
b_t	prevailing baking reward for delegating XTZ	
$B(t,T) = \prod_{s=t}^{T-1}(1+b_s)$	compounding factor in XTZ between t and T	
i_t	prevailing interest rate in the uUSD market (provided that the money is staked in the minter conversion pool)	1.550E-09/ 5.01% p.a.
$\underline{i}$	uUSD interest rate floor	1.280E-10/ 0.40% p.a.
$\bar{i}$	uUSD interest rate cap	8.192E-09/29.48% p.a.
$I(t,T) = \prod_{s=t}^{T-1}(1+i_s)$	gross compounding factor in uUSD between t and T	
τ	frequency for resetting of uUSD interest rate in seconds (weekly)	$7 \times 24 \times 3\,600$
x_t	prevailing interest rate spread owed to the platform	3.160E-10/ 1.00% p.a.
$X(t,T) = \prod_{s=t}^{T-1}(1+i_s-x_s)$	net compounding factor in uUSD between t and T	

(vi) *fees*

Variable	Description	Default Value(s)
m	minting fee to the platform	1.56%
k	transaction fee to the platform	0.00%
$c = c_1 + c_2$	total call option fee (buyback)	
c_1	call option fee to the token holder	25.00%
c_2	call option fee to the platform	0.00%
$p = p_1 + p_2$	total put option fee (gold standard)	
p_1	put option fee to the minter	6.25%
p_2	put option fee to the platform	0.00%
h	emergency step in bonus	12.50%

1.2. Interest Rate Policy.



$$i_{t+\tau} = \min\left\{\bar{i}, \max\left\{\underline{i}, i_t - \operatorname{sgn}\left(\tilde{R}_t - 1\right) \times \frac{2^{\left\lfloor \min\{|\tilde{R}_t - 1|, \overline{\Delta}\}\cdot 2^{-2}\cdot 10^2 \right\rfloor} - 1}{2^{35}}\right\}\right\}.$$

1.3. Roll-Forwards.

vault	CCY	value of collateral at time t		value of collateral one second later
j	XTZ	$C_t^{(j)}$	$\longrightarrow$	$C_{t+1}^{(j)} = C_t^{(j)} B(t, t+1)$

vault	CCY	minted coins at time t		minted coins one second later
j	uUSD	$M_t^{(j)}$	$\longrightarrow$	$M_{t+1}^{(j)} = M_t^{(j)} I(t, t+1)$

wallet	CCY	value of wallet at time t		value of wallet one second later
ℓ	uUSD	$A_t^{(\ell)}$	$\longrightarrow$	$A_{t+1}^{(\ell)} = A_t^{(\ell)} X(t, t+1)$

profit of platform	CCY	residual coins
	uUSD	$A_t^{(\ell)} x_t$

1.4. Actions, Rights & Obligations.

(i) *value chain of minting at time t in the vault j*

asset	CCY	process
underlying	USD	1
		$\downarrow$
collateral	XTZ	$\frac{1}{S_t} = \underbrace{(1-m)\frac{1}{S_t}}_{C_t^{(j)}} \underbrace{+m\frac{1}{S_t}}_{\text{minting fee}}$
		$\downarrow$
digital asset	uUSD	$\underbrace{\frac{1}{\lambda_1}(1-m)\frac{1}{S_t}}_{M_t^{(j)}}$

(ii) *transfer of α uUSD at time t between two wallets*

wallet	CCY	components at time $t-$		value of wallet at time t
ℓ_1	uUSD	$A_{t-}^{(\ell_1)} = \underbrace{A_{t-}^{(\ell_1)} - \frac{\alpha}{1-k}}_{\overset{!}{\geq}0} \underbrace{+(1-k)\frac{\alpha}{1-k}}_{\text{transfer amount}} \underbrace{+k\frac{\alpha}{1-k}}_{\text{transaction fee}}$		$A_t^{(\ell_1)} = A_{t-}^{(\ell_1)} - \frac{\alpha}{1-k}$
			$\searrow$	
ℓ_2	uUSD	$A_{t-}^{(\ell_2)}$		$A_t^{(\ell_2)} = A_{t-}^{(\ell_2)} + \alpha$

(iii) *call option of γ uUSD at time t*

vault	CCY	components at time $t-$		components at time t
j	XTZ	$C_{t-}^{(j)} = C_{t-}^{(j)} - \frac{\gamma}{S_t}(1+c) \underbrace{+\frac{\gamma}{S_t} + \frac{\gamma}{S_t}c_1}_{\text{fair value + agio}} \underbrace{+\frac{\gamma}{S_t}c_2}_{\text{option fee}}$		$C_t^{(j)} = C_{t-}^{(j)} - \frac{\gamma}{S_t}(1+c)$
			$\longrightarrow$	
	uUSD	$\underbrace{M_{t-}^{(j)}}_{\overset{!}{\geq}\gamma}$		$M_t^{(j)} = M_{t-}^{(j)} - \gamma$

wallet	CCY	components at time $t-$		components at time t
ℓ	uUSD	$\underbrace{A_{t-}^{(\ell)}}_{\overset{!}{\geq}\gamma}$		$A_t^{(\ell)} = A_{t-}^{(\ell)} - \gamma$
			$\longrightarrow$	
	XTZ	$C_{t-}^{(\ell)}$		$C_t^{(\ell)} = C_{t-}^{(\ell)} + \frac{\gamma}{S_t}(1+c_1)$

(iv) *put option of β uUSD at time t*

wallet	CCY	components at time $t-$		components at time t
ℓ	uUSD	$\underbrace{A_{t-}^{(\ell)}}_{\overset{!}{\geq}\beta}$	$\longrightarrow$	$A_t^{(\ell)} = A_{t-}^{(\ell)} - \beta$
	XTZ	$C_{t-}^{(\ell)} = C_{t-}^{(\ell)} - \frac{\beta}{S_t}p_2 \underbrace{+\frac{\beta}{S_t}p_2}_{\text{option fee}}$		$C_t^{(\ell)} = C_{t-}^{(\ell)} + \frac{\beta}{S_t}(1-p)$

vault	CCY	components at time $t-$		components at time t
j	XTZ	$C_{t-}^{(j)} = C_{t-}^{(j)} - \frac{\beta}{S_t}(1-p_1) \underbrace{+\frac{\beta}{S_t} - \frac{\beta}{S_t}p_1}_{\text{fair value - disagio}}$	$\longrightarrow$	$C_t^{(j)} = C_{t-}^{(j)} - \frac{\beta}{S_t}(1-p_1)$
	uUSD	$\underbrace{M_{t-}^{(j)}}_{\overset{!}{\geq}\beta}$		$M_t^{(j)} = M_{t-}^{(j)} - \beta$

(v) *emergency step in*

vault	CCY	components at time $t-$		components at time t
j	XTZ	$\underbrace{C_{t-}^{(j)}}_{(1+h)M_{t-}^{(j)} \overset{!}{\leq} S_t C_{t-}^{(j)} \overset{!}{\leq} \lambda_0 M_{t-}^{(j)}}$	$\longrightarrow$	$C_t^{(j)} = C_{t-}^{(j)} - \frac{1+h}{S_t} \cdot \frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}$
	uUSD	$M_{t-}^{(j)}$		$M_t^{(j)} = M_{t-}^{(j)} - \frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}$

wallet	CCY	components at time $t-$		components at time t
ℓ	uUSD	$\underbrace{A_{t-}^{(\ell)}}_{\overset{!}{\geq}\frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}}$	$\longrightarrow$	$A_t^{(\ell)} = A_{t-}^{(\ell)} - \frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}$
	XTZ	$C_{t-}^{(\ell)}$		$C_t^{(\ell)} = C_{t-}^{(\ell)} + \frac{1+h}{S_t} \cdot \frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}$

control calculation:

$$\frac{S_t C_t^{(j)}}{M_t^{(j)}} = \frac{S_t C_{t-}^{(j)} - (1+h) \cdot \frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}}{M_{t-}^{(j)} - \frac{\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}}{\lambda_1 - (1+h)}} = \frac{S_t C_{t-}^{(j)}\big(\lambda_1 - (1+h)\big) - (1+h) \cdot \Big(\lambda_1 M_{t-}^{(j)} - S_t C_{t-}^{(j)}\Big)}{M_{t-}^{(j)}\big(\lambda_1 - (1+h)\big) - \lambda_1 M_{t-}^{(j)} + S_t C_{t-}^{(j)}} = \lambda_1.$$

If $S_t C_{t-}^{(j)} < (1+h) M_{t-}^{(j)}$ and if someone is still willing to step in (presumably provided that $S_t C_{t-}^{(j)} > 1$), then the bonus reduces accordingly.

2. Model Validation

2.1. Interest Rate Policy. uUSD is supposed to replicate the time value of money in USD; hence, it should approximately hold $u_t \approx i_t$. However, uUSD coins are subject to credit risk and potential asset illiquidity. This observation gives rise to positive spreads with respect to the USD numéraire. On the other hand, negative spreads would provide an incentive for minting uUSD coins and promote the asset liquidity.

Provided that the whole ecosystem remains in a sufficiently over-collateralised state, the interest rate update, as specified above, may have a stabilising effect on the FX rate $\widetilde{R}_t$, at least to some extent. If $\widetilde{R}_t > 1$, then $i_{t+\tau} - i_t$ will be negative and depreciate the uUSD currency. Conversely, if $\widetilde{R}_t < 1$, then $i_{t+\tau} - i_t$ will be positive and, technically seen, increase the attractiveness of the uUSD currency. However, ceteris paribus, the interest rate increase could also result in a self-exciting downturn of the recovery rate due to the vicious circle of the more and more elevated credit risk. Therefore, it may be advisable to incorporate an upper limit $\bar{i}$ for the uUSD interest rates. $i_t < 0$ must be ruled out since interest is only paid out to investors who have previously staked their stable tokens in the minter conversion pool. Once the ecosystem has ended up in an under-collateralised state (e.g., due to a jump of S_t), then the interest rate policy should be disabled as it would only increase the number of outstanding coins at the cost of a reducing recovery rate. This would eliminate the likelihood of a return to full recovery. It is recommended to incorporate a flag that indicates whether the ecosystem is in a sound over-collateralised state $\{f_t \geq 1\}$ or in an under-collateralised state $\{f_t < 1\}$. In the latter case, the uUSD interest rate policy should be disabled immediately and reinstated only in the recurrence of an over-collateralised state.

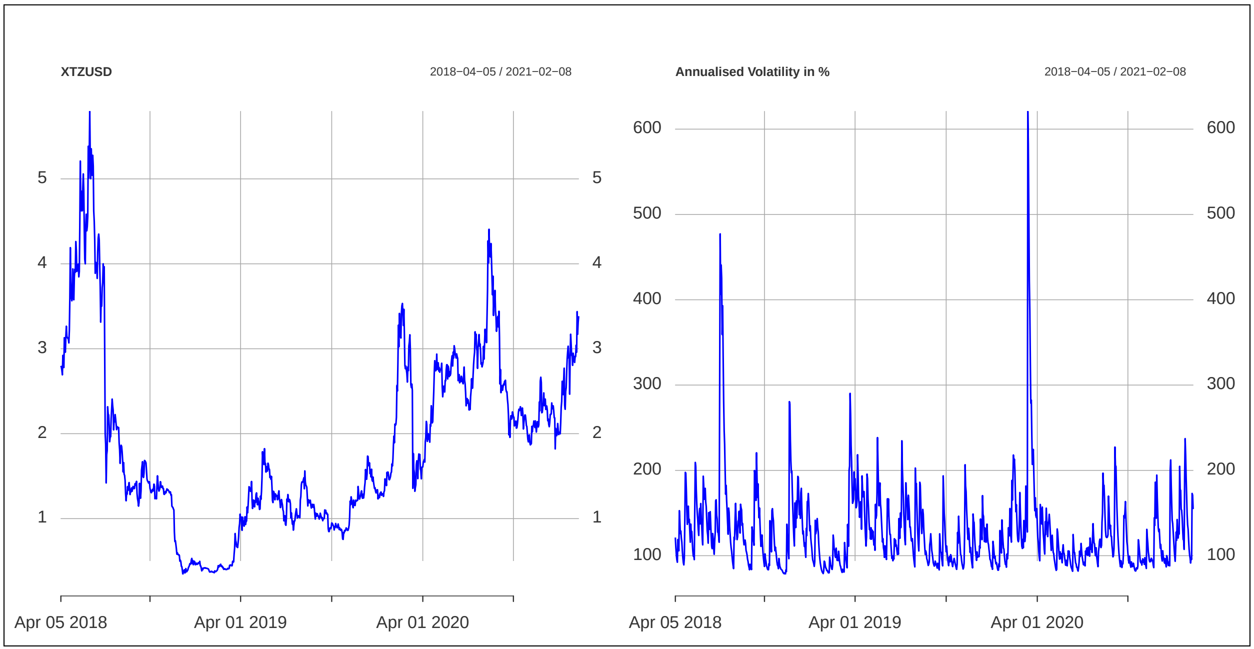
As the computation power in smart contracts should be efficient to limit gas fees, the interest rate feature of the stable tokens uses a mixed approach between simple interest rates and compounded interest rates. This friction is not material. The compounding is accounted for at least weekly. In the extreme case, if it held $i_t \equiv \bar{i}$ throughout a year, then the discrepancy between the textbook formula and the implemented mixed approach over a year would amount to $(1 + i_t)^{52 \cdot \tau} - (1 + \tau \cdot i_t)^{52} \approx 0.08\%$. In most circumstances, the discrepancy is negligible.

2.2. Stability of the FX Rate. Both the holders and minters of uUSD have the perpetual rights and obligations of an immediate conversion option (gold standard) and an immediate buyback option respectively. Let $\varepsilon > 0$.

- As long as $\widetilde{R}_t > 1 - p$ it should not be favourable to utilise the XTZ conversion right. If 1 uUSD was offered for $1 - p - \varepsilon$ USD, then the potential buyer could exercise the XTZ conversion right and request for an equivalent amount of $1 - p$ USD in XTZ, i.e., $(1 - p)/S_t$ XTZ. This leaves him or her with a riskless profit of ε USD. We tacitly assume that S_t is not too volatile during the transaction. If this inherent assumption was not satisfied, then the target range would become slightly blurrier. The conversion right is advantageous for holders of uUSD as the recovery of the token is likely not to fall below $1 - p$ USD. The conversion right is also advantageous for the minters since 1 uUSD was initially sold for 1 USD and bought back for $1 - p$ USD. However, minters must bear the market risk on the frozen collateral. If the ecosystem is in an under-collateralised state $\{f_t < 1\}$, then the put option should not comprise any fees and the payoff be f_t/S_t in XTZ; the corresponding coins would be burnt immediately. Minters would lose the converted portion of the collateral.
- As long as $R_t S_t < 1 + c$ it should not be favourable to utilise the buyback option. If 1 uUSD was demanded for $1 + c + \varepsilon$ USD, then any minter could exercise his or her buyback option for the equivalent exercise price of $1 + c$ USD in XTZ, i.e., $(1 + c)/S_t$ XTZ. After having sold on the coin for $(1 + c + \varepsilon)/S_t$ XTZ, this leaves him or her with a riskless profit of ε USD. This right is detrimental for holders of uUSD, since their coins may be converted without their consent; however, the inherent economic value of their holding increased from 1 USD to $1 + c$ USD without doing anything.

The functioning of the platform relies on a sufficient asset liquidity. One has to accept the fact that the ecosystem is not entirely controllable. The platform works with economic incentives to encourage certain user behaviour. In theory, a malicious operator with access to large funds and no intention of making money could ignore the incentive structure of the platform and create unintended outcomes (e.g., a discrepancy between $\widetilde{R}_t$ and 1).

2.3. Volatility of the Collateral (Daily Basis). Historically, the FX rate XTZUSD, denoted by S_t, was highly volatile. We calibrated a GARCH(1,1) with skewed Student-t distributed innovations. The left-hand side exhibits the historical quotes and the right-hand side the calibrated time-dependent annualised volatility.



Based on this calibrated model, we simulated 10 000 randomly initialised paths with a time horizon of five years. This corresponds to roughly 11 million data points for the estimated likelihood of a margin call over different terms.

term	$1w$	$1m$	$3m$	$6m$	$1y$	$2y$
first passage time probability of λ_0/λ_1 in %	2.85	12.84	28.20	37.28	44.80	51.10
standard deviation of the estimate in bps	0.44	0.88	1.18	1.27	1.30	1.31

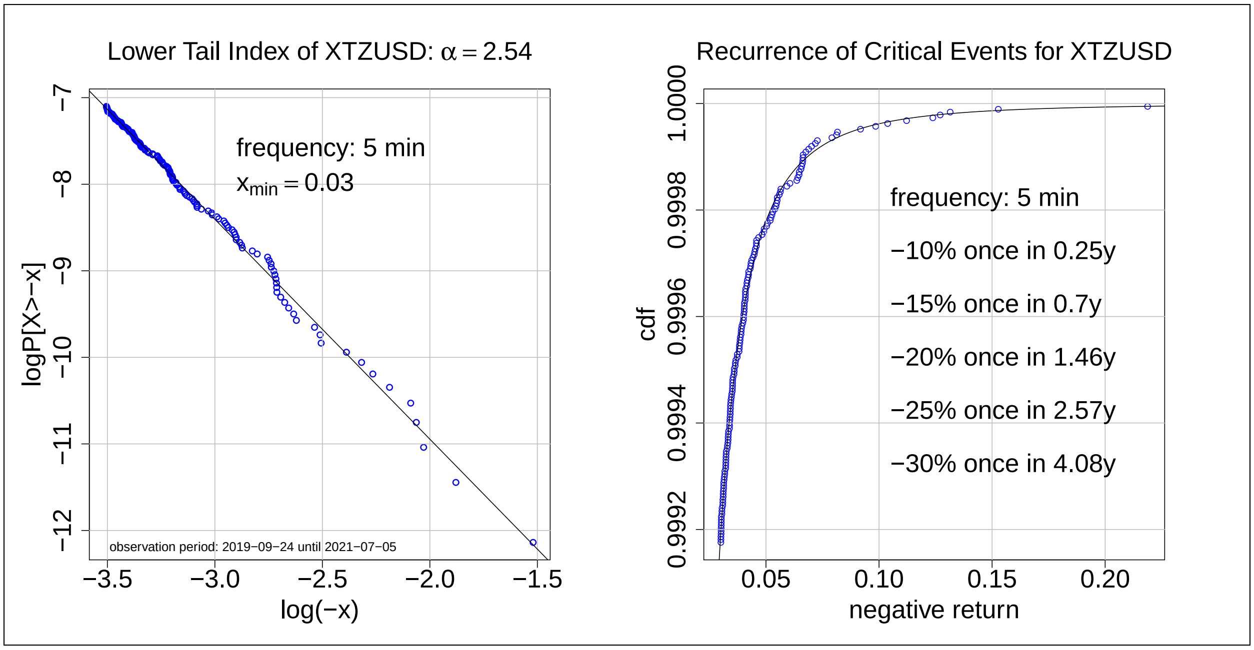
If margin calls were not honoured, then the default probability over different terms can be quantified as follows.

term	$1w$	$1m$	$3m$	$6m$	$1y$	$2y$
first passage time probability of $1/\lambda_1$ in %	0.34	1.77	7.32	15.82	27.08	38.00
standard deviation of the estimate in bps	0.15	0.35	0.68	0.96	1.16	1.27

Historically, since April 2018, there occurred no incidents with FX downturns in the region of $1/\lambda_1$ within $1w$, whereas there were three over a term of $1m$.

t	T	S_T/S_t
08-06-2018	05-07-2018	26.91%
07-11-2018	06-12-2018	25.83%
20-02-2020	16-03-2020	37.41%

2.4. Volatility of the Collateral (Intraday Basis). We estimated the lower tail index of S_t based on historical data with an observation frequency of 5 minutes. To this end, we regressed the logarithm of the absolute negative returns beyond 3% against the logarithm of the counter cumulative empirical distribution function. This resulted in a lower tail index of 2.54. Based on the calibrated Pareto distribution, we can derive the recurrence of critical events over time horizons of 5 minutes. It needs to be noted that we only have a limited historical time series. Hence, the «true» tail index might be much lower and more detrimental.



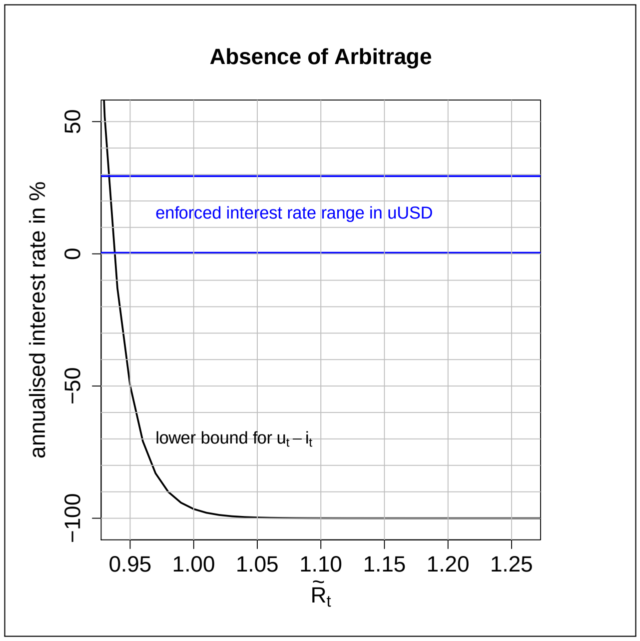
2.5. Roundtrips. Let us assume that the interest rates u_t and i_t in the USD and the uUSD market respectively have been fixed between the time instances $[t, t+\tau]$. The investment of 1 USD in the domestic and the synthetic market over $1w$ respectively yield the final wealth levels $(1+u_t)^\tau$ and

$$\frac{\widetilde{R}_{t+\tau}}{\widetilde{R}_t}(1+i_t)^\tau\,.$$

Due to the conversion and the buyback option (and, as the case may be, further measures), one establishes $\widetilde{R}_{t+\tau} \in [1-p, 1+c]$. Therefore, one can preclude obvious arbitrage opportunities in terms of the constraints

$$\begin{aligned} \frac{1-p}{\widetilde{R}_t}(1+i_t)^\tau \leq \quad & (1+u_t)^\tau \quad \leq \frac{1+c}{\widetilde{R}_t}(1+i_t)^\tau \\ \iff \qquad \left(\sqrt[\tau]{\frac{1-p}{\widetilde{R}_t}}-1\right)(1+i_t) \leq \quad & u_t - i_t \quad \leq \left(\sqrt[\tau]{\frac{1+c}{\widetilde{R}_t}}-1\right)(1+i_t)\,. \end{aligned}$$

The upper bound is irrelevant as it beyond 1E15% p.a. for realistic parameter choices. The lower bound starts to matter if $\widetilde{R}_t$ gets closer and closer to the lower bound of the target range. However, it is doubtful whether the critical states can be reached at all. Increasing interest rate levels i_t supposedly have a stabilising effect. They move the FX rate $\widetilde{R}_t$ away to levels, where the lower bound is no longer problematic.



2.6. Further Observations. For the stability of the ecosystem, it is recommended to monitor the concentration risk of the collateral.

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